Risk-averse structural topology optimization under random fields using stochastic expansion methods

Jesús Martínez-Frutos¹*, David Herrero-Pérez¹, Mathieu Kessler², Francisco Periago¹,²

¹Computational Mechanics and Scientific Computing Group, Technical University of Cartagena, Campus Muralla del Mar, 30202 Cartagena, Murcia, Spain
²Department of Applied Mathematics and Statistics, Technical University of Cartagena, Campus Muralla del Mar, 30202 Cartagena, Murcia, Spain

Abstract

This work proposes a level-set based approach for solving risk-averse structural topology optimization problems considering random field loading and material uncertainty. The use of random fields increases the dimensionality of the stochastic domain, which poses several computational challenges related to the minimization of the Excess Probability as a measure of risk awareness. This problem is addressed both from the theoretical and numerical viewpoints. First, an existence result under a typical geometrical constraint on the set of admissible shapes is proved. Second, a level-set continuous approach to find the numerical solution of the problem is proposed. Since the considered cost functional has a discontinuous integrand, the numerical approximation of the functional and its sensitivity combine an adaptive anisotropic Polynomial Chaos (PC) approach with a Monte-Carlo (MC) sampling method for uncertainty propagation. Furthermore, to address the increment of dimensionality induced by the random field, an anisotropic sparse grid stochastic collocation method is used for the efficient computation of the PC coefficients. A key point is that the non-intrusive nature of such an approach facilitates the use of High Performance Computing (HPC) to alleviate the computational burden of the problem. Several numerical experiments including random field loading and material uncertainty are presented to show the feasibility of the proposal.

Keywords: Risk-averse topology optimization; Random fields; Material uncertainty; Stochastic expansion methods; Existence theory; Level-set method.

*Corresponding author

Email addresses: jesus.martinez@upct.es (Jesús Martínez-Frutos¹), david.herrero@upct.es (David Herrero-Pérez¹), mathieu.kessler@upct.es (Mathieu Kessler²), f.periago@upct.es (Francisco Periago¹,²)

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1. Introduction

Topology optimization aims at finding the optimal distribution of material within a design domain such that an objective functional is minimized under certain constraints [1]. Contrary to size and shape optimization methods, topology optimization permits to obtain a material distribution without assuming any prior structural configuration. This provides engineering designers with a powerful tool to find innovative and high-performance conceptual designs at the early stages of the design process. Such a technique has been successfully applied to improve the design of complex industrial problems, such as aeronautical, aerospace and naval applications [2]. However, deterministic conditions are usually assumed, and thus obviating the different sources of uncertainty which may affect not only the safety and reliability of structures but also their performance. Robustness and reliability have been recognized as a desirable property in structural systems and have sparked considerable interest in the scientific and engineering communities. The need for including uncertainty quantification and propagation stages during the design process has shown to be a key issue for solving real-world engineering problems in several fields, such as aeronautical and aerospace [3], civil [4], automotive [5] and mechanical [6] engineering, to name but a few. This fact, together with the development of probabilistic uncertainty propagation methods, has fostered the interest for considering uncertainty within the topology optimization problems, giving rise to the formulation of several approaches embraced under the term of Topology Optimization Under Uncertainty (TOUU) methods.

Such methods can be broadly classified, according to the representation and treatment of uncertainties, into non-probabilistic and probabilistic approaches. The former methods [7] do not require the statistical information about the uncertainty of the phenomenon but a qualitative notion about its magnitude. The worst-case approach [8, 9, 10], taking the form of a min-max optimization problem, and fuzzy techniques [11], making use of fuzzy set theory, are methods included in this category. The main drawback of these approaches is that they are often too conservative, due to overestimation of uncertainty, and may lead to optimal designs with poor structural performance. The latter methods make use of the probabilistic characterization of the uncertainty of the phenomenon. Several formulations have been proposed in this context to address the wide concept of “structural robustness”. These formulations differ from each other in the design objective as well as in the way the uncertainty is incorporated within the optimization formulation. Robust Topology Optimization (RTO) incorporates the first two statistical moments of the cost functional to obtain optimal designs which are less sensitive to variations in the input data [12, 13, 14, 15, 16]. Reliability-Based Topology Optimization (RBTO) aims at determining the best design solution (with respect to prescribed criteria such as stiffness, weight or construction costs) while explicitly considering the unavoidable effects of uncertainty. This is done by posing the constraints in terms of the probability of constraint violation (probability of failure) [17, 18, 19]. On the other hand, Risk-Averse Topology Optimization (RATO) [20, 21, 22] does not aims at mini-
mizing a deterministic prescribed criteria but a risk function that quantifies the expected loss related with the damages (e.g. excess probability) [23]. Whereas RBTO provides optimal designs in terms of a deterministic prescribed criteria with enough reliability level, RATO provides the best design from the point of view of risk-aversion.

One of the main challenges of TOUU methods is the computational burden of addressing the problem, which still remains even though significant numerical and theoretical advances have been achieved in the last years. A key point of the computational challenge is that when the solution of the underlying Partial Differential Equation (PDE) is expensive, one can only afford to solve a few hundred samples. This is far from the required number of samples for estimating a probability. Such a drawback is exacerbated in high-dimensional stochastic domains, such as those obtained with random fields. This work is concerned with this issue in the context of RATO problems under random fields.

The RATO problem is introduced by the early work of Conti et al. [21] proposing a risk-averse approach in the finite dimensional context of shape optimization. The uncertainty is introduced in the applied loads, and hence, there is a linear dependence of the system response with respect to uncertainty. Making use of the assumption of linear elasticity and quadratic functional cost, the applied loads can be expressed as a linear combination of a small number of deterministic basis loads and a set of uncertain coefficients, which are represented by means of discrete random variables. The main advantage of this approach is that the computational cost scales linearly in the number of linearly independent applied loads. Allaire and Dapogny [24] addressed a similar problem under the assumption of small uncertainties using a first-order Taylor expansion of the cost functional with respect to the uncertainties. Similar ideas are used in [25] for the case of geometrical uncertainties. These approaches lead to deterministic approximations of the cost functional with similar computational burden to that of multiple load problems, where the number of loads is similar to the finite number of random variables. Existence of solution for worst-case optimization problems has been obtained, e.g. in [26, 27]. However, this issue is presented in [20] as an open problem for RATO. The early work of Conti et al. [21] is recently extended to introduce a new class of stochastic shape optimization problems by means of the concept of stochastic dominance [28]. Rather than handling risk aversion in the functional cost, dominance constrains are considered for bounding the Excess Probability of the design shape with that of a given reference shape. This issue, is the main difference with respect to other formulations which consider probabilistic constraints such as RBTO. We refer the interested reader to [28] for more details on this passage. To the best knowledge of the authors, these are the only works dealing with RATO problem for the system of linear elasticity. Such articles are numerical in nature and no existence results of optimal shapes have been proved so far.

In this paper, the topology optimization problem of risk awareness for continuous elastic structures with random field uncertainties in material properties and applied loads is analyzed. In contrast to previous works in RATO, the evaluation of Excess Probability functional under random field material and/or
loading uncertainties requires integrations over failure regions, which represents a key difficulty. This is not the case of the RTO problem considered in our previous work [13], where the integrands in the cost functionals are smooth with respect to the random variable. This permits an efficient use of stochastic collocation methods to approximate the integrals in the random domain that arise in such a problem. In this work, the problem is addressed both theoretically and numerically. From a theoretical point of view, existence of a solution under volume and geometrical constraints on the set of admissible designs is proved. The proof of this result is obtained by adapting the one in [13] to the case in which the integrand is not continuous but lower semi-continuous. From a numerical point of view, the problem is solved by adapting the level-set method to this probabilistic framework. In addition to the high computational cost, which is inherent to optimization problems with uncertainties represented by random fields, the main difficulty lies in the fact that the integrand in the cost functionals measuring risk awareness is discontinuous. Similarly to [21], a smooth approximation of this functional is introduced. Then, the continuous gradient of the approximated functional is explicitly computed. Although at the theoretical level both the approximated cost functional and its gradient are smooth, at the numerical level they are not. Hence, the numerical approximation, by using stochastic collocation methods (as the one considered in [13]), of the corresponding integrals in the random domain may lead to incorrect results if a few collocations points are located in the unknown discontinuities. The use of very fine stochastic grids is unaffordable due to the high computational burden. This difficulty is addressed in this work by the use of an adaptive anisotropic Polynomial Chaos (PC) expansion and anisotropic sparse grid method for uncertainty propagation in combination with a Monte-Carlo (MC) method for the numerical approximation of the cost functional and its sensitivity. Stochastic expansion methods have been previously applied in RTO to estimate statistical moments [16, 29, 30], and they have shown recently their effectiveness in more complex problems, such as in RBTO to estimate failure probabilities and their sensitivities [31, 32]. This work complements the previous studies [21, 32] by investigating the use of adaptive anisotropic PC approaches to efficiently solve RATO problems under random field uncertainties. Finally, the proposed approach is applied to several two-dimensional benchmarks where uncertainties are incorporated in loads and material properties along with different probability distribution types and random fields.

The paper is organized as follows. Section 2 is devoted to the problem statement including the proper formulation of the RATO problems as well as the proof of the existence result under volume and geometrical constraints on the set of admissible designs. The proposed numerical resolution of RATO problems for continuous structures using the level-set method is presented in section 3. Section 4 is devoted to the uncertainty modeling and the anisotropic PC approach used for the numerical integration over failure regions. Some numerical experiments are presented in Section 5 to verify the effectiveness of the proposed development. Such experiments include uncertainty in loads and material properties represented by random fields. Finally, some conclusion and
concluding remarks are presented in section 6.

2. Problem statement

2.1. The state law

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space, and let \(\mathcal{O} \subset \mathbb{R}^d\) \((d = 2 \text{ or } 3 \text{ in applications})\) be a bounded Lipschitz domain whose boundary \(\partial\mathcal{O}\) is decomposed into three disjoint parts

\[
\partial\mathcal{O} = \Gamma_D \cup \Gamma_N \cup \Gamma_0, \quad |\Gamma_D| > 0,
\]

where \(|\cdot|\) stands for the Lebesgue measure. Consider the system of linear elasticity with random input data

\[
\begin{aligned}
-\text{div}(Ae(u(x, \omega))) &= f \quad \text{in } \mathcal{O} \times \Omega, \\
ue(x, \omega) &= 0 \quad \text{on } \Gamma_D \times \Omega, \\
(Ae(u(x, \omega))) \cdot n &= g \quad \text{on } \Gamma_N \times \Omega, \\
(Ae(u(x, \omega))) \cdot n &= 0 \quad \text{on } \Gamma_0 \times \Omega,
\end{aligned}
\]

where \(e(u) = \frac{1}{2} (\nabla u^T + \nabla u)\) is the strain tensor, \(n\) is the unit outward normal vector to \(\partial\mathcal{O}\), and \(A\) is the material Hooke’s law, defined for any symmetric matrix \(\zeta\) by

\[
A \zeta = 2\mu \zeta + \lambda (Tr\zeta) Id,
\]

where \(\lambda = \lambda(x, \omega)\) and \(\mu = \mu(x, \omega)\) are the Lamé moduli of the material, which may depend on both \(x = (x_1, \ldots, x_d) \in \mathcal{O}\) and \(\omega \in \Omega\). In the same vein, the volume and surface loads \(f\) and \(g\) depend on a spatial variable \(x\) and on a random event \(\omega \in \Omega\), i.e. \(f = f(x, \omega)\) and \(g = g(x, \omega)\). The divergence (div) and the gradient \(\nabla\) operators in (2) involve only derivatives with respect to the spatial variable \(x\).

Since the domain \(\mathcal{O}\) is changing during the optimization process, \(\lambda, \mu\) and \(f\) must be known for all possible configurations of \(\mathcal{O}\). Thus, a working bounded domain \(D\) is introduced, which contains all admissible domains \(\mathcal{O}\) and satisfies that \(\Gamma_D \cup \Gamma_N \subset \partial D\). Therefore, it is assumed that \(\lambda, \mu\) and \(f\) are defined in the bounded working domain \(D\). \(\Gamma_N\) is kept fixed during optimization. Hence, \(g(\omega) \in L^2(\Gamma_N)^d\) a.s. \(\omega \in \Omega\).

From now on, if \(X\) is a Banach space and \(1 \leq p \leq \infty\), \(L^p_p(\Omega; X)\) denotes the Lebesgue-Bochner space composed of all (equivalence classes of) strongly measurable functions \(h: \Omega \to X\) whose norm

\[
\|h\|_{L^p_p(\Omega; X)} = \begin{cases} 
(\int_\Omega \|h(\cdot, \omega)\|_X^p \, d\mathbb{P}(\omega))^{1/p}, & p < \infty, \\
\text{ess sup}_{\omega \in \Omega} \|h(\cdot, \omega)\|_X, & p = \infty
\end{cases}
\]

is finite. In order for problem (2) to be well-posed, the following assumptions on the uncertain input parameters of system (2) are considered:
(A1) $\lambda(x,\omega), \mu(x,\omega) \in L^\infty_0(\Omega; L^\infty(D))$ and there exist positive constants $\mu_{\text{min}}, \mu_{\text{max}}, \lambda_{\text{min}}, \lambda_{\text{max}}$ such that

$$0 < \mu_{\text{min}} \leq \mu(x,\omega) \leq \mu_{\text{max}} < \infty \quad \text{a.e. } x \in D, \quad \text{a.s. } \omega \in \Omega,$$

and

$$0 < \lambda_{\text{min}} \leq 2\mu(x,\omega) + d\lambda(x,\omega) \leq \lambda_{\text{max}} < \infty \quad \text{a.e. } x \in D, \quad \text{a.s. } \omega \in \Omega,$$

(A2) $f = f(x,\omega) \in L^2(\Omega; L^2(D)^d)$,

(A3) $g = g(x,\omega) \in L^2(\Omega; L^2(\Gamma_N)^d)$.

To introduce an appropriate functional framework for problem (2), the Hilbert space

$$V_O = \{ v \in H^1(O) : v|_{\Gamma_D} = 0 \text{ in the sense of traces} \},$$

equipped with the usual $H^1(O)$-norm, is considered.

A weak solution of (2) is a random field $u \in L^2(\Omega; V_O)$ such that

$$\int_{\Omega} \int_{O} A e(u(x,\omega)) \cdot e(v(x,\omega)) \, dx \, d\mathbb{P}(\omega) = \int_{\Omega} f \cdot v \, dx + \int_{\Gamma_N} g \cdot v \, ds \, d\mathbb{P}(\omega), \quad \forall v \in L^2(\Omega; V_O).$$

(3)

A straightforward application of the Lax-Milgram lemma allows one to state the well-posedness of (2). Moreover, there exists $c > 0$ such that

$$\| u \|_{L^2(\Omega; V_O)} \leq c \left( \| f \|_{L^2(\Omega; L^2(O)^d)} + \| g \|_{L^2(\Omega; L^2(\Gamma_N)^d)} \right)$$

and

$$\| u(\omega) \|_{V_O} \leq c \left( \| f(\omega) \|_{L^2(O)^d} + \| g(\omega) \|_{L^2(\Gamma_N)^d} \right), \quad \text{a.s. } \omega \in \Omega. \quad (4)$$

Notice that, thanks to assumption (A1), the constant $c = c(O, \mu_{\text{min}}, \lambda_{\text{min}})$ in (4) does not depend on $\omega \in \Omega$.

2.2. The set of admissible shapes

The $\varepsilon$-cone property, introduced by Chenais [33], is presented below. Let a point $y \in \mathbb{R}^d$, a vector $\xi \in \mathbb{R}^d$ and $\varepsilon > 0$ be given. Consider the cone

$$C(y, \xi, \varepsilon) = \{ z \in \mathbb{R}^d : (z - y, \xi) \geq \cos \varepsilon |z - y|, \quad 0 < |z - y| < \varepsilon \}.$$

An open set $O \subset \mathbb{R}^d$ is said to have the $\varepsilon$-cone property if for all $x \in \partial O$ there exists a unit vector $\xi_x$ such that $\forall y \in \overline{O} \cap B(x, \varepsilon)$, the property $C(y, \xi_x, \varepsilon) \subset O$ holds.

Consider the class

$$O_\varepsilon = \{ O \text{ open, } O \subset D, \quad O \text{ has the } \varepsilon\text{-cone property} \}. \quad 6$$
For instance, the bounded, open and convex sets are some of the domains in the class $\mathcal{O}_\varepsilon$ for $\varepsilon$ small enough [34, Prop. 2.4.4]. Moreover, if $\mathcal{O}$ satisfies the $\varepsilon$-cone property, then $\partial \mathcal{O}$ is uniformly Lipschitz, where the geometrical constants which are involved in the uniform Lipschitz character of $\mathcal{O}$ depend only on $\varepsilon$. And conversely, a uniform Lipschitz domain $\mathcal{O}$ satisfies the $\varepsilon$-cone property for a positive $\varepsilon$ which depends on the constants bounding the uniform Lipschitz character of $\mathcal{O}$ (see the proof of [34, Th. 2.4.7 and Remark 2.4.8]). Imposing a volume constraint $V_0$ on an admissible domain, the class of admissible domains is

$$U_{ad} = \{ \mathcal{O} \in \mathcal{O}_\varepsilon, \quad \Gamma_D \cup \Gamma_N \subset \partial \mathcal{O}, \quad |\mathcal{O}| = V_0 \}, \quad \text{with } 0 < V_0 < |D|.$$ 

The class of admissible shapes is composed of open and bounded sets $\mathcal{O}$ of fixed volume $V_0$, i.e. $|\mathcal{O}| = V_0 > 0$, where $|\cdot|$ stands for the Lebesgue measure. In addition, the boundary of $\mathcal{O}$ satisfies the condition (1) with $\Gamma_D$ and $\Gamma_N$ fixed so that $\Gamma_0$ is the part of the boundary to be optimized.

### 2.3. Formulation and existence of solution of the Risk-Averse Topology Optimization problem

For each realization $\omega \in \Omega$ and for a given admissible domain $\mathcal{O}$, consider the compliance

$$J(\mathcal{O}, \omega) = \int_{\mathcal{O}} f(x, \omega) \cdot u_{\mathcal{O}}(x, \omega) \, dx + \int_{\Gamma_N} g(x, \omega) \cdot u_{\mathcal{O}}(y, \omega) \, ds,$$

where $u_{\mathcal{O}} = u_{\mathcal{O}}(x, \omega)$ is a weak solution of (2).

Throughout the paper, if $X \in L^1(\Omega)$ is a real-valued random variable, its expected value is denoted by

$$\mathbb{E}(X) = \int_{\Omega} X(\omega) \, d\mathbb{P}(\omega).$$

The variance is widely used as a measure of deviation in the context of RTO [12, 14, 35]. This, however, could not be the best choice in some minimization problems because the variance treats the excess over the mean equally as the shortfall. This is the case when one is interested in minimizing the expected loss related with damages caused by catastrophic failures [20]. Accordingly, for a given threshold value $\eta \in \mathbb{R}$, the following cost functional (which is typically referred to as Excess Probability [21] in the context of RATO) is considered:

$$J_{EP_\eta}(\mathcal{O}) = \mathbb{P} \{ \omega \in \Omega : J(\mathcal{O}, \omega) > \eta \} = \mathbb{E}(H(J(\mathcal{O}, \omega) - \eta))$$

where

$$H(z) = \begin{cases} 
0, & z \leq 0 \\
1, & z > 0
\end{cases}$$
is the Heaviside function. The following optimization problem is investigated in this paper:

$$ (P_{EP_n}) \min \{ J_{EP_n}(\mathcal{O}) : \mathcal{O} \in \mathcal{U}_{ad} \}. $$

It is well known that if an admissible domain is assumed to be only Lebesgue measurable the deterministic shape optimization problem may be ill-posed due to the lack of closure of the set of admissible domains with respect to the weak-${\ast}$ topology in $L^\infty(D;\{0,1\})$. To overcome this difficulty, additional conditions, such as smoothness, geometrical or topology constraints, are imposed on the class of admissible designs. In this work, the standard $\varepsilon$-cone assumption [33] is considered. To prove the existence of a solution for $(P_{EP_n})$, the set of admissible domains $\mathcal{U}_{ad}$ is equipped with an appropriate topology so that the compactness of $\mathcal{U}_{ad}$ and the continuity of $J_{EP_n}(\mathcal{O})$ are ensured. Following [34, Definition 2.2.3], let $\{\mathcal{O}_n\}_{n \geq 1}$ and $\mathcal{O}$ be (Lebesgue) measurable sets of $\mathbb{R}^d$. It is said that $\{\mathcal{O}_n\}$ converges to $\mathcal{O}$ in the sense of characteristic functions as $n \to \infty$ if

$$ 1_{\mathcal{O}_n} \to 1_{\mathcal{O}} \text{ in } L^p_{loc}(\mathbb{R}^d) \quad \forall p \in [1,\infty], $$

where $1_B$ stands for the characteristic function of a measurable set $B \subset \mathbb{R}^d$.

The class of domains $\mathcal{O}_\varepsilon$ enjoys the following compactness property: let $\{\mathcal{O}_n\}_{n \geq 1}$ be a sequence in the class $\mathcal{O}_\varepsilon$. Then, up to a subsequence, still labelled by $n$, $\mathcal{O}_n$ converges to some $\mathcal{O}^* \in \mathcal{O}_\varepsilon$, in the sense of characteristic functions (also in the sense of Hausdorff distance and in the sense of compacts sets). We refer the readers to [34, Th. 2.4.10] for more details on this passage.

Another very important property that the domains in the class $\mathcal{O}_\varepsilon$ satisfy is the following uniform extension property [33, Th. II.1]: there exists a positive constant $K$, which depends only on $\varepsilon$, such that for all $\mathcal{O} \in \mathcal{O}_\varepsilon$ there exists a linear and continuous extension operator

$$ P_\mathcal{O} : H^1(\mathcal{O}) \to H^1(\mathbb{R}^d), \quad \text{with } \|P_\mathcal{O}\| \leq K. \quad (7) $$

We are now in a position to prove the following existence result:

**THEOREM 2.1.** Problem $(P_{EP_n})$ is well posed, i.e., it has, at least, one solution.

**Proof.** Let $\mathcal{O}_n \subset \mathcal{U}_{ad}$ be a minimizing sequence. Due to the compactness property of $\mathcal{O}_\varepsilon$ mentioned above there exists $\mathcal{O}^* \in \mathcal{O}_\varepsilon$ and a subsequence, still labelled by $n$, such that $\mathcal{O}_n$ converges to $\mathcal{O}^*$. Moreover, thanks to the convergence in the sense of characteristic functions, $\mathcal{O}^*$ satisfies the volume constraint and hence $\mathcal{O}^* \in \mathcal{U}_{ad}$. The domain $\mathcal{O}^*$ is the candidate to be the minimizer we are looking for.

Similarly to [13, Th. 2.1], the following point-wise (w.r.t. $\omega \in \Omega$) convergence holds:

$$ J(\mathcal{O}_n,\omega) \to J(\mathcal{O}^*,\omega) \text{ as } n \to \infty, \text{ a.s. } \omega \in \Omega. \quad (8) $$

8
Indeed, let $u_n(x, \omega)$ and $u_{\mathcal{O}^*}(x, \omega)$ be the solutions to (2) associated to $\mathcal{O}_n$ and $\mathcal{O}^*$, respectively. Denoting by $\hat{u}_n(\cdot, \omega) := P_{\mathcal{O}_n}(u_n)(\cdot, \omega)$ the extension of $u_n(\cdot, \omega)$ to $\mathbb{R}^d$, by (4) and (7) one has

$$\|\hat{u}_n(\omega)\|_{V_D} \leq \|P_{\mathcal{O}_n}\|_{V_{\mathcal{O}_n}} \leq Ke \left( \|f(\omega)\|_{L^2(D)^d} + \|g(\omega)\|_{L^2(\Gamma_N)^d} \right),$$

a.s. $\omega \in \Omega$, which proves that $\hat{u}_n(\omega)$ is bounded in $V_D$. Thus, by extracting a subsequence, it converges weakly in $V_D$, and strongly in $L^2(D)^d$ to some $u^*(\omega) \in V_D$. Let us prove that

$$u^*|_{\mathcal{O}^*}(\omega) = u_{\mathcal{O}^*}(\omega), \quad \omega \in \Omega.$$  

From the definition of $u_n$, it follows that

$$\int_{\mathcal{O}_n} Ae(u_n) : e(v) \, dx = \int_{\mathcal{O}_n} f \cdot v \, dx + \int_{\Gamma_N} g \cdot v \, dx \quad \forall v \in V_D, \text{ a.s. } \omega \in \Omega,$$

which is equivalent to

$$\int_D 1_{\mathcal{O}_n} Ae(\hat{u}_n) : e(v) \, dx = \int_D 1_{\mathcal{O}_n} f \cdot v \, dx + \int_{\Gamma_N} g \cdot v \, dx \quad \forall v \in V_D, \text{ a.s. } \omega \in \Omega.$$  

The weak convergence of $\hat{u}_n$ in $V_D$ and the strong convergence in $L^2(D)$ of the characteristic functions permit to take the limit to get

$$\int_D 1_{\mathcal{O}^*} Ae(u^*) : e(v) \, dx = \int_D 1_{\mathcal{O}^*} f \cdot v \, dx + \int_{\Gamma_N} g \cdot v \, dx \quad \forall v \in V_D, \text{ a.s. } \omega \in \Omega,$$

and also, thanks to the extension property (7),

$$\int_{\mathcal{O}^*} Ae(u^*) : e(v) \, dx = \int_{\mathcal{O}^*} f \cdot v \, dx + \int_{\Gamma_N} g \cdot v \, dx, \quad \forall v \in V_{\mathcal{O}^*}.$$  

This proves that $u^*|_{\mathcal{O}^*}(\omega) = u_{\mathcal{O}^*}(\omega)$ a.s. $\omega \in \Omega$. Since this is valid for any subsequence, the whole sequence $\hat{u}_n$ converges to $u_{\mathcal{O}^*}$. Let us consider the compliance in the form

$$J(\mathcal{O}_n, \omega) = \int_{\mathcal{O}_n} f \cdot u_n \, dx + \int_{\Gamma_N} g \cdot u_n \, ds = \int_D 1_{\mathcal{O}_n} f \cdot \hat{u}_n \, dx + \int_{\Gamma_N} g \cdot \hat{u}_n \, ds.$$

Since $1_{\mathcal{O}_n} f \rightarrow 1_{\mathcal{O}^*} f$ strongly in $L^2(D)$ and $\hat{u}_n \rightharpoonup u^*$ weakly in $V_D$, the limit in this expression can be taken to obtain (8).

Since $F_n(\omega) := H(J(\mathcal{O}_n, \omega) - \eta)$ are integrable and non-negative, by Fatou’s lemma,

$$\int_{\Omega} \liminf_{n \to \infty} F_n(\omega) \, d\mathcal{P}(\omega) \leq \liminf_{n \to \infty} \int_{\Omega} F_n(\omega) \, d\mathcal{P}(\omega). \quad (9)$$

By (8) and the lower semi-continuity of the Heaviside function,

$$H(J(\mathcal{O}^*, \omega) - \eta) \leq \liminf_{n \to \infty} F_n(\omega),$$

9
Integrating in both sides of this expression and using (9) yields
\[
\int_{\Omega} H ( J ( O^\ast, \omega ) - \eta ) \, d\mathbb{P}(\omega) \leq \int_{\Omega} \liminf_{n \to \infty} F_n(\omega) \, d\mathbb{P}(\omega) \\
\quad \leq \liminf_{n \to \infty} \int_{\Omega} F_n(\omega) \, d\mathbb{P}(\Omega).
\]
Since $O_n$ is a minimizing sequence, the result follows. \square

**Remark 1.** We notice that the $\varepsilon$-cone condition in the class of admissible designs has been introduced in order to obtain the preceding existence result. As is usual, $\varepsilon$-cone condition is not taking into account in the numerical resolution approach, which is presented in the next section.

### 3. Numerical resolution of $(P_{EP_n})$ via the level-set method

The numerical resolution of $(P_{EP_n})$ is addressed by extending the deterministic level-set method, as introduced in [36], to the probabilistic setting described in the preceding sections. This is a gradient-based algorithm and hence only local minimizers may be computed. For the sake of completeness, we start by recalling the notion of shape derivative.

#### 3.1. A brief review on shape derivative

The concept of shape derivative was introduced by Hadamard [37]. Here we follow the approach of Murat and Simon [38]. Other classical references on this topic are [39, 34, 40].

Given a reference domain $O$, we consider perturbations of this domain of the form $O_\theta = (I + \theta)(O)$, where $I$ is the identity operator and $\theta \in W^{1, \infty}(\mathbb{R}^d; \mathbb{R}^d)$. It is known that, for $\theta$ small enough, $(I + \theta)$ is a diffeomorphism.

Now let $J(O)$ be a (shape) functional. The shape derivative of $J(O)$ is defined as the Fréchet derivative in $W^{1, \infty}(\mathbb{R}^d; \mathbb{R}^d)$ at 0 of the mapping $\theta \to J(O_\theta)$, i.e.,

\[
J(O_\theta) = J(O) + J'(O)(\theta) + o(\theta), \quad \text{with} \quad \lim_{\theta \to 0} \frac{|o(\theta)|}{\|\theta\|} = 0,
\]

where $J'(O)$ is a continuous linear form on $W^{1, \infty}(\mathbb{R}^d; \mathbb{R}^d)$.

Hadamard’s structure theorem states that $J'(O)(\theta)$ depends only on the normal trace $\theta \cdot n$ on the boundary. In particular, the following result is very useful to compute shape derivatives. See [39, Propositions 6.22 and 6.24] for a proof.

**Theorem 3.1.** Let $O$ be a smooth bounded open domain. Let $F_1(x) \in W^{1,1}(\mathbb{R}^d)$ and $F_2(x) \in W^{2,1}(\mathbb{R}^d)$. The shape functional

\[
I(O) = \int_{O} F_1(x) \, dx + \int_{\partial O} F_2(x) \, ds
\]
is differentiable at $\mathcal{O}$ and its shape derivative is given by

$$I'(\mathcal{O}) (\theta) = \int_{\partial \mathcal{O}} \theta \cdot n \left( F_1 + \frac{\partial F_2}{\partial n} + \text{div}(n) \cdot F_2 \right) \, ds,$$  \hspace{1cm} (11)

where $\text{div}(n)$ is the mean curvature of $\partial \mathcal{O}$.

3.2. Explicit (formal) computation of the continuous shape derivative

The volume constraint, as given in the definition of the admissible shapes, is incorporated in the functional cost through an augmented Lagrangian method. According to [41], the augmented Lagrangian function is constructed as

$$J_{EP_\vartheta}^L (\mathcal{O}) = \int_\Omega H \left( J(\mathcal{O}, \omega) - \eta \right) \, d\mathbb{P}(\omega) + L_1 \left( \int_\mathcal{O} dx - V_0 \right) + \frac{1}{2} L_2 \left( \int_\mathcal{O} dx - V_0 \right)^2,$$  \hspace{1cm} (12)

being $L_1$ and $L_2$ the Lagrange multiplier and the penalty parameter, respectively. The Lagrange multiplier is updated at each iteration $n$ according to

$$L_1^{(n+1)} = L_1^n + \frac{1}{L_2} \left( \int_\mathcal{O} dx - V_0 \right).$$  \hspace{1cm} (13)

The penalty parameter $L_2$ is updated as $L_2^{n+1} = \alpha L_2^n$, where $\alpha \in (0, 1)$. The penalty parameter is updated until a minimum value is reached.

Since the cost functional $J_{EP_\vartheta}^L (\mathcal{O})$ has a discontinuous integrand, the following smooth approximation (see, e.g., [22]) is considered

$$J_{EP_\vartheta}^{L,\epsilon} (\mathcal{O}) = \int_\Omega \left( 1 + e^{-\frac{\epsilon}{2} (J(\mathcal{O}, \omega) - \eta)} \right)^{-1} \, d\mathbb{P}(\omega) + L_1 \left( \int_\mathcal{O} dx - V_0 \right) + \frac{1}{2} L_2 \left( \int_\mathcal{O} dx - V_0 \right)^2,$$  \hspace{1cm} (14)

with $0 < \epsilon \ll 1$.

**Remark 2.** The choice of the parameter $\epsilon$ is a delicate issue. If the value of $\epsilon$ is too high, the accuracy of the Heaviside approximation is degraded. On the contrary, if the value of $\epsilon$ is too small the values of the sensitivities may be close to zero when a few number of sampling points are taken in the transition regions, i.e. transitions between zero and non-zero probability regions. Following the guidelines indicated in [31], the parameter $\epsilon$ is adaptively updated during the optimization in the numerical experiments. This is done by selecting the value of $\epsilon$ as a function of the standard deviation of the cost functional.

Let $\theta \in W^{1,\infty} (\mathbb{R}^d; \mathbb{R}^d)$ be the deformation domain vector field. Since $\partial \mathcal{O} = \Gamma_D \cup \Gamma_N \cup \Gamma_0$, where $\Gamma_D$ and $\Gamma_N$ are fixed and $\Gamma_0$ is the part of the boundary to be optimized, the vector field $\theta$ vanishes on $\Gamma_D \cup \Gamma_N$. 


Indeed, a straightforward computation yields
\begin{equation}
\left( J_{\mathbb{E}P_2}^{L_\epsilon} \right)^* (\mathcal{O})(\theta) = \int_0^1 \theta \cdot n \left[ \int_\Omega C_1(\omega) \left( 2f \cdot u - Ae(u) : e(u) \right) d\mathbb{P}(\omega) \right] ds + L_1 \int_{\Gamma_0} \theta \cdot n ds + \frac{1}{\epsilon^2} \left( \int_\Omega dx - V_0 \right) \int_{\Gamma_0} \theta \cdot n ds,
\end{equation}
where the random variable $C_1(\omega)$ is given by
\begin{equation}
C_1(\omega) = 2 \left( 1 + e^{-\frac{\epsilon}{2}(J(\mathcal{O},\omega)-\eta)} \right)^{-2} e^{-\frac{\epsilon}{2}(J(\mathcal{O},\omega)-\eta)}.
\end{equation}

Sketch of the proof. Although the computation of the shape derivative of (14) is standard, a formal proof by using the Lagrangian method due to Céa [42] is presented below for the sake of completeness.

To begin with, the Lagrangian
\begin{equation}
\mathcal{L}(\mathcal{O}, \hat{u}, \hat{p}) = \int_\Omega \left( 1 + e^{-\frac{\epsilon}{2}(J_\mathcal{O} f \cdot \hat{u} dx + f_{\Gamma_N} g \cdot \hat{u} ds - \eta)} \right)^{-1} d\mathbb{P}(\omega)
\end{equation}
which is defined for $\hat{u}, \hat{p} \in L_2^\mathcal{E} (\Omega; H^1(\mathbb{R}^d, \mathbb{R}^d))$ such that $\hat{u} = \hat{p} = 0$ on $\Gamma_D$, is considered. For a given $\mathcal{O}$, let $(u, p)$ be a stationary point of $\mathcal{L}$. Equating to 0 the partial derivative of $\mathcal{L}$ with respect to $\hat{u}$ in the direction $v \in L_2^\mathcal{E} (\Omega; V_\mathcal{O})$ evaluated at $(\mathcal{O}, u, p)$ yields the adjoint equation of the optimization problem. Indeed, a straightforward computation yields
\begin{equation}
0 = \left< \frac{\partial \mathcal{L}}{\partial u} (\mathcal{O}, u, p), v \right> \quad \text{or}
\end{equation}
\begin{equation}
= \int_\Omega C_1(\omega) \left( \int_\Omega f \cdot v dx + \int_{\Gamma_N} g \cdot v ds \right) d\mathbb{P}(\omega)
\end{equation}
where $C_1(\omega)$ is given by (16). After integrating by parts and by taking $v$ with compact support in $\mathcal{O} \times \Omega$ gives
\begin{equation}
-\text{div}(Ae(p)) = -C_1(\omega)f \quad \text{in } \mathcal{O} \times \Omega.
\end{equation}
Similarly, varying the trace of $v(\cdot, \omega)$ on $\Gamma_N$ and on $\Gamma_0$ yields the boundary conditions
\begin{equation}
Ae(p) \cdot n = -C_1(\omega)g \quad \text{on } \Gamma_N \times \Omega
\end{equation}
and
\begin{equation}
Ae(p) \cdot n = 0 \quad \text{on } \Gamma_0 \times \Omega.
\end{equation}
Therefore, $p \in L_2^\mathcal{E} (\Omega; V_\mathcal{O})$ solves the adjoint equation
\begin{equation}
\begin{cases}
-\text{div}(Ae(p(x, \omega))) = -C_1(\omega)f(x, \omega) & \text{in } \mathcal{O} \times \Omega, \\
p(x, \omega) = 0 & \text{on } \Gamma_D \times \Omega,
\end{cases}
\end{equation}
\begin{equation}
\begin{cases}
(Ae(p(x, \omega))) \cdot n = -C_1(\omega)g(x, \omega) & \text{on } \Gamma_N \times \Omega, \\
(Ae(p(x, \omega))) \cdot n = 0 & \text{on } \Gamma_0 \times \Omega.
\end{cases}
\end{equation}
and consequently $p(x, \omega) = -C_1(\omega)u(x, \omega)$.

A direct computation shows that $< \frac{\partial L}{\partial \hat{p}} (O, u, p), v > = 0$ reduces to the direct problem (2).

Next, since $u = u(\Omega)$ solves (2), for $\hat{p} \in L_2^2(\Omega; H^1(\mathbb{R}^d; \mathbb{R}^d))$ such that $\hat{p} = 0$ on $\Gamma_D$, one has

$$L(\Omega, u, \hat{p}) = \int_{\Omega} \left( 1 + e^{-\epsilon} \left( \int_{\Omega} f^u dx + \int_{\Gamma_N} g u ds - \eta \right) \right)^{-1} dP(\omega).$$

Differentiating this expression with respect to $\hat{p} = \hat{p}$ in the direction $\theta$ and applying the chain rule theorem yields

$$\left( \int_{\Omega} \left( 1 + e^{-\epsilon} \left( \int_{\Omega} f^u dx + \int_{\Gamma_N} g u ds - \eta \right) \right)^{-1} dP(\omega) \right)'(\theta) = \frac{\partial L}{\partial \hat{p}} (\Omega, u, \hat{p})(\theta) + < \frac{\partial L}{\partial u} (\Omega, u, \hat{p}), u'(\Omega) >.$$

Taking $\hat{p} = p$, solution to the adjoint system (22), the last term vanishes. Thus, the first term in the right-hand side of (15) is then formally obtained by computing the partial derivative of $L(\Omega, u, p)$ with respect to $\hat{p}$ in the direction $\theta$, by using (11), and by taking into account that $\theta = 0$ on $\Gamma_D \cup \Gamma_N$.

Finally, by using again formula (11) and the chain rule theorem one easily gets the third and fourth terms in the right-hand side of (15) associated to $L_1$ and $L_2$.

\[ \square \]

**Remark 3.** Although in this paper we focus on the compliance objective function (5), other types of cost functionals may be treated in a similar way. For instance, another typical choice is

$$J_2(\Omega, \omega) = \left( \int_{\Omega} \kappa(x) |u(x, \omega) - u_0|^\alpha dx \right)^{1/\alpha},$$

(23)

which represents a least square error compared to a target displacement. Here, $\alpha \geq 2$, $u_0 \in L^\alpha(D)$ is a prescribed target, and $\kappa \in L^\infty(D)$ is a non-negative weighting factor. In particular, up to the introduction of a suitable adjoint equation, the shape derivative of (a smooth approximation of) the cost functional $\hat{J}_2(\omega) = \int_{\Omega} H(J_2(\Omega, \omega) - \eta) dP(\omega)$ may be computed by using the same arguments as in the preceding result.

3.3. Description of the numerical algorithm

Following an original idea by Osher and Sethian [43], which was imported to shape optimization in [36, 44], the unknown domain $\Omega$ is described through the zero level set of a function $\psi = \psi(t, x)$ defined on the working domain $D$ as follows

$$\forall x \in D, \forall t \in (0, T), \begin{cases} \psi(x, t) < 0 & \text{if } x \in \Omega, \\ \psi(x, t) = 0 & \text{if } x \in \partial \Omega, \\ \psi(x, t) > 0 & \text{if } x \notin \Omega. \end{cases}$$

(24)
Here, $t$ stands for a fictitious time that accounts for the step parameter in the descent algorithm. Hence, the domain $\mathcal{O} = \mathcal{O}_t$ is evolving in time during the optimization process. In the numerical context, the level-set function $\psi$ is discretized at the vertices of a simplicial mesh of $D$. During the optimization process, $\psi$ solves the Hamilton-Jacobi equation

$$\frac{\partial \psi}{\partial t} + (\theta \cdot n) |\nabla \psi| = 0, \quad t \in (0, T), \; x \in D,$$

Equation (25), which is complemented with Neumann type boundary conditions, is solved in the hole domain $D\setminus\mathcal{O}$ by using the so-called “ersatz material” approach. Precisely, the hole $D\setminus\mathcal{O}$ is filled by a weak phase that mimics the void, but at the same time avoids the singularity of the rigidity matrix. Thus, an elasticity tensor $A^*(x)$ which equals $A$ in $\mathcal{O}$ and $10^{-3} \cdot A$ in $D\setminus\mathcal{O}$ is introduced. Decomposing the boundary of $D$ as $\partial D = \partial D_0 \cup \Gamma_D \cup \Gamma_N$, the displacement field $u(x, \omega)$ solves

$$\begin{cases}
-\text{div}(A^* e(u(x, \omega))) = f & \text{in } D \times \Omega, \\
u(x, \omega) = 0 & \text{on } \Gamma_D \times \Omega, \\
(A^* e(u(x, \omega))) \cdot n = g & \text{on } \Gamma_N \times \Omega, \\
(A^* e(u(x, \omega))) \cdot n = 0 & \text{on } \partial D_0 \times \Omega,
\end{cases}$$

(26)

The chosen descent direction in (25) is $\theta = -v n$, with

$$v = \int_{\Omega} C_1(\omega) (2f \cdot u - Ae(u) : e(u)) \, d\mathbb{P}(\omega) + L_1 + \frac{1}{\epsilon^2} \left( \int_{\Omega} dx - V_0 \right),$$

(27)

which has been previously computed in Proposition 3.1. Note that since the vector displacement $u$ is defined throughout the domain $D$, the velocity field $v$ is also defined in $D$ and not only on the boundary $\partial \mathcal{O}$. As it is usual in this context, the equation (25) is solved using an explicit first order upwind scheme. To avoid singularities, the level-set function is periodically reinitialized. The readers are referred to [36] for more details on this passage.

The crucial step is the numerical approximation of the advection velocity (27), which requires the numerical computation of the random state $u(x, \omega)$, i.e. the solution to (26). It is important to emphasize that although the random variable $C_1(\omega)$, which appears in (27), is (theoretically) smooth with respect to the random event $\omega$, for $\epsilon$ small, numerically it is not. The same problem appears when evaluating the cost functional $J_{L,\epsilon}^{EP_\eta}(\mathcal{O})$ at each iteration of the descent algorithm. As a consequence, stochastic collocation methods are not the best choice to compute those integrals and it is more convenient to use a Monte Carlo (MC) sampling strategy. However, the use of a direct MC method requires the numerical resolution of (2) at a large number of sampling points $\omega_k \in \Omega$ and at each step of the descent method. This makes a direct MC method unaffordable from a computational point of view. For this reason, and similarly to [31], the MC method is combined with a Polynomial Chaos (PC) method for uncertainty propagation. More precisely, $u(x, \omega)$ is approximated with a PC expansion method. Then, the MC method is applied to this approximation to compute the integrals in the random domain which appear in the cost functional $J_{L,\epsilon}^{EP_\eta}(\mathcal{O})$, and in its sensitivity. Next section provides details on all these issues.
4. Numerical integration over failure regions using anisotropic polynomial chaos expansion

This section is devoted to the numerical integration over the failure regions. A key point is the fact that the integrand in the risk averse formulation is not continuous, which prevents the use of the stochastic collocation method (well-suited for smooth problems). In order to deal with this fact, MC methods can be used at the cost of increasing the computational burden significantly. This computation challenge is addressed by the combination of MC method (used for numerical integration) and a PC expansion (used as an emulator) for approximating the smooth solution of the elasticity system. In addition, the coefficients in the PC expansion are computed with an anisotropic, non-intrusive, stochastic collocation method, which can take advantage of the parallelizable character of collocation methods.

4.1. Uncertainty modeling

The uncertainties acting in the PDE input data with spatial variations are characterized by random fields. In order to address numerically the problem, the random fields are discretized according to the following assumption:

(A4) **Finite dimensional noise:** the random input data of (2), i.e. Lamé coefficients and loads, depend on a finite number \( N \) of real-valued random variables \( \{Y_n\}_{n=1}^N \).

A typical choice to fulfill the assumption (A4) is the truncated K-L expansion [45] of a second-order random field. Indeed, let \( U : D \times \Omega \to \mathbb{R} \) be such that \( U \in L_2^2(\Omega; L_2^2(D)) \). Denote by \( \bar{U}(x) := \mathbb{E}(U(x, \cdot)) = \int_D U(x, \omega) \, d\mathbb{P}(\omega) \) the mean function of \( U \), and by

\[
C(x, x') := \mathbb{E} \left[ (U(x, \cdot) - \bar{U}(x, \cdot))(U(x', \cdot) - \bar{U}(x', \cdot)) \right]
\]

its covariance function. Then, the random field \( U(x, \omega) \) admits the so-called Karhunen-Loève (K-L) expansion

\[
U(x, \omega) = \bar{U}(x) + \sum_{n=1}^{\infty} \sqrt{\gamma_n} b_n(x) Y_n(\omega)
\]

where \( Y_n \) are pairwise uncorrelated random variables with zero mean and unit variance, and \( \gamma_n \) and \( b_n \) are the eigenvalues and eigenvectors, respectively, of the compact and self-adjoint operator

\[
\psi \mapsto \int_D C(x, x') \psi(x') \, dx', \quad \psi \in L_2^2(D).
\]

If the random field is Gaussian, then \( Y_n \sim N(0, 1) \) are independent and identically distributed standard Gaussian variables. The sum in (28) converges in \( L_2^2(\Omega; L_2^2(D)) \). For numerical simulation purposes, the series (28) is always
truncated at some large enough term \( N \), i.e., the random field \( U(x, \omega) \) is approximated as

\[
U(x, \omega) \approx U_N(x, \omega) = U(x) + \sum_{n=1}^{N} \sqrt{\gamma_n} b_n(x) Y_n(\omega).
\]  

(29)

The convergence rate of \( U_N \) depends on the decay of the eigenvalues \( \gamma_n \), which in turn depend on the smoothness of its associated covariance function. See [46] and the references therein for more details.

The representation (29) is well-suited to represent uncertainty in the loads \( f \) and \( g \). In the case of a surface load \( g \), as in the problem under consideration, the spatial domain for the random field is \( \Gamma_N \). However, since Lamé coefficients are positive in nature, uncertainty in material properties cannot be represented by a truncated K-L expansion of a Gaussian field. Instead, log-normal random fields are considered in this paper, which take the form

\[
a(x, \omega) \approx a_N(x, \omega) = \exp (\eta + \xi U_N(x, \omega)), \quad a = \lambda \text{ and/or } \mu,
\]  

(30)

where \( U_N \) is as in (29), and \( \eta, \xi \) are location and scale parameters. Yet, from the theoretical point of view, Gaussian variables cannot be used for modeling uncertainty of Lamé coefficients. This is due to the lower and upper uniform boundedness condition imposed in (A1). However, in practice, a Gaussian variable may be accurately approximated considering its truncation at some large enough lower and upper points. In addition, some regularity on the eigenfunctions \( b_n(x) \) should be imposed. In the case of Gaussian random fields with a covariance function continuous in \( D \times D \), the eigenfunctions \( b_n(x) \) are continuous and bounded in \( D \times D \). We again refer the readers to [46] for more details.

4.2. Anisotropic non-intrusive PC expansion for uncertainty propagation

Since the random variables \( \{Y_n\}_{n=1}^{N} \) are mapped from the sample space \( \Omega \) to \( \mathbb{R} \), by using a standard localization argument, the variational formulation of (2) can be rewritten as

\[
\int_D A e(u(x, y)) : e(\varphi(x)) \, dx = \int_D f(x, y) \cdot \varphi(x) \, dx + \int_{\Gamma_N} g(x, y) \cdot \varphi(x) \, ds, \quad \forall \varphi \in V_D, \quad \rho - a.e. \text{ in } \Lambda,
\]  

(31)

where denoting \( \Lambda_n = Y_n(\Omega) \) with \( 1 \leq n \leq N \), \( \Lambda = \prod_{n=1}^{N} \Lambda_n \subset \mathbb{R}^N \) and \( \rho = \rho(y) \) the joint probability density function of \( Y = (Y_1, \cdots, Y_N) \), which is assumed to both exist and to factorize as \( \rho(y) = \prod_{n=1}^{N} \rho_n(y_n), \quad y = (y_1, \cdots, y_n) \in \Lambda \). The solution \( u = u(x, y) \) to (31) belongs to the space \( L^2_p(\Lambda; V_D) \), which is defined as

\[
L^2_p(\Lambda; V_D) := \left\{ h : \Lambda \to V_D, \quad \int_{\Lambda} \|h(y)\|^2_{V_D} \rho(y) \, dy < \infty \right\}.
\]

The reader is referred to [46, Chapter 9] for more details on this passage.
Following [31, 47, 48], let \( \{ \psi_p (y_n) \}_{p=1}^{\infty} \) be an orthonormal basis of the space \( L^2_{\rho_n} (\Lambda_n) := \left\{ \hat{h} : \Lambda_n \to \mathbb{R}, \int_{\Lambda_n} |\hat{h}(y_n)|^2 \rho_n(y_n) \, dy_n < \infty \right\} \) composed of a suitable class of orthonormal polynomials. Since \( L^2_{\rho} (\Lambda) = \bigotimes_{n=1}^{N} L^2_{\rho_n} (\Lambda_n) \), a multivariate orthonormal polynomial basis of \( L^2_{\rho} (\Lambda) \) is constructed as

\[
\{ \psi_p (y) = \prod_{n=1}^{N} \psi_{p_n} (y_n) \}_{p=(p_1, \cdots, p_N) \in \mathbb{N}^N}.
\]

Let now \( \ell \in \mathbb{N}_+ \) be an index indicating the level of approximation in the random space \( L^2_{\rho} (\Lambda) \).

A primary drawback of PC expansion is the well known curse of dimensionality, according to which the number of unknown coefficients increases as the space of random variables becomes of higher dimension. In this context, adaptive sparse PC expansion [49] has been proposed to significantly reduce the computational cost. Additionally, the number of terms required by the K-L expansion in (29) can lead to unaffordable computational problem when full tensor product rules are used to compute the PC coefficients. Note that if the same number of points \( R \) is used in each direction, then the total number of points in a full tensor rule is \( R^N \). To address this issue, the isotropic sparse grid approaches [50, 51, 52] permit to reduce the number of collocation points keeping the level of accuracy for problems whose random variables weight equally in the solution. However, the convergence rate is deteriorated for highly anisotropic problems [52]. This is the case of K-L expansions in which random variables corresponding to large eigenvalues have more influence than those associated to low eigenvalues. In this case, an anisotropic sparse tensor product can lead to highly accurate solutions while reducing drastically the computational requirements. In this work, an adaptive anisotropic PC approach is presented to reduce the computational cost involved in RATO problem with random fields.

This method is composed of the following main steps: (i) based on the decay properties of the eigenvalues of the K-L expansion, a vector of weights \( \beta \) is chosen (see (33)). (ii) Both \( \beta \) and the level \( \ell \) determine a multi-index set (32), which is used to select the degree of polynomials in the different stochastic directions. (iii) From this multi-index, a finite dimensional space (34) approximating \( L^2_{\rho} (\Lambda) \) is introduced. (iv) Since the space \( L^2_{\rho} (\Lambda; V_O) \), where the solution to (31) lives in, is isomorphic to \( V_O \otimes L^2_{\rho} (\Lambda) \), the solution to (31) is expressed as in (35), where the coefficients in this expansion are computed by using an anisotropic stochastic collocation method. Next, details on these issues are described.

Given a vector of weights \( \beta = (\beta_1, \cdots, \beta_N) \in \mathbb{N}^N_+ \) for the different stochastic directions, the multi-index set

\[
I_{\beta} (\ell, N) = \left\{ p = (p_1, \cdots, p_N) \in \mathbb{N}^N_+ : \sum_{n=1}^{N} p_n \beta_n \leq \ell \beta_{min} \right\}, \quad (32)
\]

with \( \beta_{min} = \min_{1 \leq n \leq N} \beta_n \) is considered. The optimal choice of \( \beta \) depends on the analyticity properties of the solution of (31) with respect to the parameter
\(y \in \Lambda\). For the case of Gaussian random fields the \(n\)-th term of \(\beta\) is taken as

\[
\beta_n = \frac{1}{2\sqrt{2\pi n} \|b_n\|_{L^\infty(D)}},
\]

(33)

where \((\gamma_n, b_n(x))\) is the eigenpair in the K-L expansion (29).

Finally, the anisotropic approximation multivariate polynomial space is

\[
P_{I_\beta(\ell,N)}(\Lambda) = \text{span} \{ \psi_p(y), \; p \in I_\beta(\ell,N) \}.
\]

(34)

The space \(P_{I_\beta(\ell,N)}(\Lambda)\) is referred in the literature as to Anisotropic Total Degree polynomial space \([47]\). Thus, an approximated solution \(u_\ell(x,y) \in V_\partial \otimes P_{I_\beta(\ell,N)}\) of (31) is expressed in the form \([53, 54]\)

\[
u = u(x,y) \approx u_\ell(x,y) = \sum_{p \in I_\beta(\ell,N)} \hat{u}_p(x) \psi_p(y), \; \hat{u}_p \in V_\partial,
\]

(35)

where due to the orthonormality of \(\{\psi_p(y)\}_{p \in I_\beta(\ell,N)}\),

\[
\hat{u}_p(x) = \int_\Lambda u_\ell(x,y) \psi_p(y) \rho(y) \, dy \approx \int_\Lambda u(x,y) \psi_p(y) \rho(y) \, dy.
\]

(36)

This latter integral is numerically approximated using the anisotropic sparse grid collocation method described in \([13]\) considering similar multi-index sets and the weights to those given in (32) and (33), respectively. The level of quadrature for the collocation method is fixed to \(\ell + 1\), which provides enough accuracy to integrate the polynomials with order \(2p\) from (36). The anisotropic sparse grid collocation method leads to a set of uncoupled deterministic sub-problems collocated at some stochastic nodes \(y^k \in \Lambda\). The shape of the unknown domain \(\partial\) for each sub-problem is constrained to a bounding box \(D\) tessellated using a fixed grid of Lagrange \(Q_1\) finite elements. From this point on, \(V^h_D\) denotes a finite element space approximating the Sobolev space \(V_D\) introduced in Subsection 2.1, where \(h\) stands for the size of the spatial mesh.

It remains to analyze how the level of approximation \(\ell\) is chosen. Since the main goal in this paper is to minimize the cost functional \(J_{EP_t,h}^{L,c}(\partial)\) and its derivative, the level of approximation \(\ell\) is adaptively chosen so as to comply with a prescribed accuracy level \(\delta\) for those quantities. This is done as follows:

(i) Compute the vector of weights \(\beta\) according to (33).

(ii) Initialize the level \(\ell = 1\) and a positive, large enough, integer \(\ell\).

(iii) Compute an enriched solution \(u^h_\ell(x,y)\) of (31) by following the method described above. This enriched solution (which plays the role of exact solution) is then used to obtain an approximation \(J_{EP_t,h}^{L,c}(\partial)\) of \(J_{EP_t,h}^{L,c}(\partial)\) by using MC method, where random samplings \(y^k\) are applied to \(u^h_\ell(x,y)\).

The domain \(\partial\) used in this computation is the optimal shape of the deterministic problem - deterministic meaning that the random input parameters of (2) are replaced by their nominal (or mean) values.
Finally, the level $\ell$ is linearly increased (from $\ell = 1$ to $\ell = \ell_{opt} < \bar{\ell}$) up to the stopping criterion

$$D_{KL}(r_\ell || r_\bar{\ell}) := \int_\Lambda r_\ell(y) \log \left( \frac{r_\ell(y)}{r_\bar{\ell}(y)} \right) dy \leq \delta, \quad (37)$$

where $r_\ell$ and $r_\bar{\ell}$ are the Probability Density Functions (PDFs) of the approximation and the enriched value of $J(\mathcal{O}, \omega)$, respectively. The probability density at $y$ is computed using Kernel Density Smoothing as:

$$r_\ell(y) = \sum_{i=1}^{n_{MC}} w_i K_\varpi(y - s_i) \quad (38)$$

where $s_i$ are the values in the $n_{MC}$ Monte Carlo sample, $K_\varpi(y)$ is the zero-centered kernel function with bandwidth $\varpi$, and $w_i = 1/n_{MC}$ is the weight of each point. In this work a Gaussian kernel, which is implemented through the “ksdensity” function of MATLAB, is used:

$$K_\varpi(y - s_i) = \frac{1}{\sqrt{2\pi}\varpi} e^{-\frac{1}{2} \left( \frac{y - s_i}{\varpi} \right)^2}. \quad (39)$$

Following the Silverman’s rule [55] the bandwidth is $\varpi = \sigma_{MC} 1.06 n_{MC}^{-1/5}$, being $\sigma_{MC}$ the standard deviation of the Monte Carlo sample. $D_{KL}(r_\ell || r_\bar{\ell})$ is the so-called Kullback-Leibler divergence [56], which is a measure of how one probability distribution diverges from a second expected probability distribution. Since $D_{KL}$ provides a criterion to check how accurate the approximation is in terms of PDFs, it is appropriate in our specific situation of the functional $J^{L, \epsilon, h}_{EP_\eta}(\mathcal{O})$. In the case that the stopping criterion (37) does not hold for any $\ell < \bar{\ell}$, then a larger $\bar{\ell}$ is taken to ensure that the enriched solution $u^h_{\bar{\ell}}(x, y)$ is good enough. The whole process is then repeated taking the new value of $\bar{\ell}$ as reference.

The cost functional $J^{L, \epsilon, h}_{EP_\eta}(\mathcal{O})$ and its sensitivity, especially the constant $C_1(\omega)$, as given by (16), are numerically approximated using MC integration. The MC sampling in the random domain $\Lambda$ is applied to the approximated solution $u_\ell(x, \cdot) \in L^2(\Lambda)$ for which the explicit representation (35) is available. At the beginning of the optimization process, the number of MC sampling points is adaptively increased up to fulfill the convergence criterion (37).

**Remark 4.** The use of stochastic collocation methods during the optimization process can notably increase the computational cost. However, these methods are considered embarrassingly parallel because they have no shared state and can be executed in complete isolation. For that reason, stochastic collocation methods become an attractive tool for uncertainty propagation tasks with the advent of High Performance Computing (HPC). Task-level parallelism is used for implementing the resolution of the state equation at each $y^k \in \Lambda$ stochastic node.
in clustering. This is done by adopting a Master-Worker Pattern, which permits to perform simultaneous processing across multiple machines or processes via a Master and multiple Workers. The Master creates a set of tasks at each $y^k \in \Lambda$, which consists of the resolution of the state equation using finite element analyses. The Workers calculate these works in clustering, taking work after work until the set of tasks has been calculated. After receiving all samples at the stochastic nodes, the Master evaluates the statistical moments involved in the objective function and the shape derivative.

5. Numerical experiments

The performance of the proposal for RATO of continuous structures is evaluated solving three numerical benchmarks previously used in the context of RTO. The benchmarks have been adapted to RATO problems including random field loading and material uncertainty. The first benchmark is used to investigate the accuracy and efficiency of the proposal through a simple example using one random variable. This simple benchmark permits us to carry out a convergence study of the calculation of the PDF obtained using non-intrusive adaptive PC expansion approach and MC method. Additionally, it aims to show the effect of the prescribed threshold $\eta$ and the loading uncertainty on the optimal design of structures. The second benchmark aims to investigate the performance of the proposal when random fields are used to characterize the loading uncertainty. The last benchmark permits us to evaluate the proposal in the presence of random loads and spatially varying random material properties.

All the benchmarks make use of a fixed grid of linear quadrangular Lagrange $Q_1$ finite elements with elasticity modulus $E = 1$ (in the third experiment $E$ will be a random field with mean equal to 1) and Poisson’s ratio $\nu = 0.3$ considering plane stress conditions. The fixed grid approximation, also referred to as “ersatz material” approach [36], classifies the elements as inside $I$, outside
and boundary $B$ elements according to their position with respect to the zero-level of level set function $\psi$ (See [57] for more details). The Hooke’s law is then modified as \((\chi_e + \Delta(1 - \chi_e))A\), where $\Delta = 10^{-3}$ and $\chi_e = V_e^I/V_e$ is the volume ratio between the volume of element $V_e^I$ enclosed by $O$ and the total volume of element $V_e$. The level set function $\psi(t,x)$ is discretized at the nodes of such a fixed grid.

For all the experiments, the parameter $\epsilon$, used in the approximation of the cost functional, is updated during the optimization. According to [31], a reasonable choice consists in the selection, at iteration $n$, of $\epsilon_n$ value depending on the mean or standard deviation of the cost functional. In this vein, a value of $\epsilon = 0.01\sigma(J_{L,\epsilon,h}^{\eta,\ell}(O_n))$ is used in this work, being $\sigma(J_{L,\epsilon,h}^{\eta,\ell}(O_n))$ the standard deviation of the approximated cost at level $\ell$ associated to the design $O_n$, which is obtained at iteration $n$. Figure 1 shows that the choice of $\epsilon$ following this criterion yields reasonable accuracy for the three benchmarks. The coefficients of the PC expansion are computed using the anisotropic sparse grid scheme presented in [13] with a quadrature level $\ell+1$ and the weights given in (33). Since the considered random variables in the examples are Gaussian, a non-nested quadrature rule, whose collocation nodes are determined by the roots of Hermite polynomials, are used.

The Lagrange multiplier and the penalty parameter of the augmented Lagrangian functions [13, 31] are initialized heuristically according to

$$L_1 = 0.1 \cdot J_{L,\epsilon,h}^{\eta,\ell}(O)/V_0 \quad \text{and} \quad L_2 = \frac{1}{\text{abs}(0.01 \cdot L_1/V_0)},$$

where $O$ is the initial design. The penalty parameter is augmented, after every five iterations, multiplying its previous value by 0.8 until a minimum value of $10^{-3}$ is reached.

In all the numerical experiments the gradient descent algorithm stops at the first $n \in \mathbb{N}$ for which the two following conditions are satisfied:

$$\max_{1 \leq i \leq 5} \{|J_{L,\epsilon,h}^{\eta,\ell}(O_n) - J_{L,\epsilon,h}^{\eta,\ell}(O_{n-i})| \leq 0.01 \cdot J_{L,\epsilon,h}^{\eta,\ell}(O_n)$$

and

$$|V^n - V_0| \leq 0.005.$$

The resolution of the direct and adjoint state equations at each stochastic node is computed using a cluster with 8 nodes. Such nodes are equipped with 2 Intel Xeon E5-2620 at 2 GHz and 32 GB of RAM memory.

5.1. Numerical Experiment 1: Beam-to-cantilever design

The beam-to-cantilever problem consists of the shape optimization of a two-dimensional cantilever under uncertain loading conditions. Such a benchmark is widely used [10, 14, 58] to show the effects of uncertain loading in the robust optimal design. The left edge of the cantilever is anchored and a unit force with uncertainty in direction, centered in the horizontal line, is applied at the middle
of the right edge. The design domain is a 1x2 rectangle which is tessellated using a 64x128 regular mesh of quadrangular elements \((h_x = h_y = 0.0156)\).

The uncertain loading, the boundary conditions and the tessellation of the design domain are depicted in Figure 2(a), whereas the initialization of the level-set function is shown in Figure 2(b). The configuration and the numerical resolution of the RATO problem is as follows. The volume constraint is set to the 30% of the initial design domain. The direction \(\phi\) of the unit load follows a Gaussian distribution centered at the horizontal line with mean \(\mu_\phi = 0\) and standard deviation \(\sigma_\phi = \pi/12\).

A convergence study of the PDF obtained using non-intrusive adaptive PC expansion (PCE) method and MC simulation is performed. The convergence for the MC approach is depicted in Figures 3(a) and Figures 4(a), where the level of the PCE is \(\ell = 10\) and the number of samples of MC simulation \((n_{\text{samples}})\) is varied from \(10^2\) to \(10^6\). Figure 3(b) shows the different PDFs obtained by varying the level \(\ell\) of the PC expansion from 2 to 4. In this case, the number of samples of MC simulation is fixed to \(10^6\) and the PCE with level \(\ell = 10\) is considered as a reference. The differences depicted in Figure 3 are quantified by means of the Kullback-Leibler divergence (37) considering a level \(\ell = 10\) for the MC method and a level \(\ell = 10\) and number of samples \(10^6\) as a reference for the PCE method. The results, depicted in Figure 4, reveal that a value of \(D_{KL} = 1.56 \times 10^{-4}\) is obtained for a level \(\ell = 4\) of PCE method. In the light of these results, the integrals involved in the cost and sensitivities are evaluated using non-intrusive PCE method with level \(\ell = 4\).

The deterministic optimization required 86 iterations to converge whereas the risk-averse optimization (for case \(\eta = 10\)) required 63 iterations to converge.
Figure 3: Beam-to-cantilever problem (case $\eta = 5$): PDF of compliance calculated (a) using PCE-based MC method with different sets of samples and (b) using PCE method with different levels $\ell$.

Figure 4: The beam-to-cantilever problem: Kullback-Leibler divergence for the calculation of PDF using (a) MC method and (b) an adaptive PCE with level $\ell = 10$ considered as reference.
Deterministic $\eta = 5$ $\eta = 6$ $\eta = 7$ $\eta = 9$ $\eta = 10$

(a) (b) (c) (d) (e) (f)

Figure 5: The beam-to-cantilever problem: (a) DTO design and (b-f) RATO designs for different $\eta$ values.

Each iteration of the risk-averse optimization required to solve four different occurrences of the state equation, which were performed concurrently at each worker of the cluster. The resolution of the linear system of equations of elasticity was performed using a conjugate gradient method preconditioning with a V-cycle geometric multigrid [59, 60] using six levels and a damping factor for Jacobian smoothing $w = 0.4$. The time per iteration of the risk-averse optimization using four workers was 56.5 ms, which is slightly greater than the time required by the deterministic optimization 53.7 ms.

The Deterministic Topology Optimization (DTO) design (classical beam) for the horizontal ($\phi = 0$) unit-load is shown in Figure 5(a), whereas the RATO designs are shown from Figure 5(b) to Figure 5(f) for various magnitudes of the parameter $\eta$. One can observe that the resulting designs incorporate stiffness in the vertical direction where the uncertain loading is introduced. The risk is firstly minimized by means of topology modifications, increasing the redundancy by adding paths to transmit the load to the support, and then by thickening the cantilever members. It is worth mentioning that such a behavior is very similar to the one observed by the authors in [13] by using a RTO formulation.

Table 1 shows the probability of exceeding different magnitudes of compliance for deterministic and risk-averse designs. The following key conclusions can be drawn from the information detailed in such a Table. Firstly, the probability of exceeding a critical value $\eta_i$ achieves its minimal value when the quantity $P\{J > \eta_i\}$ is considered as objective function. This is the expected result and the Excess Probability for each RATO design is highlighted in bold. Secondly, the deterministic design is not always the worst choice when compared to risk-averse designs with $\eta \neq \eta_i$. This is attributed to the shape of the PDF resulting from the minimization of the Excess Probability. Figure 6 shows the PDF for the DTO design and diverse RATO designs using different values of $\eta$. One can observe that the mean of PDFs is higher as the prescribed threshold $\eta_i$ increases. Although the probability of exceeding $\eta_i$ is reduced, more realiza-
Table 1: Excess Probability of optimal design for different values of $\eta$ and for DTO in beam-to-cantilever benchmark.

<table>
<thead>
<tr>
<th>Case</th>
<th>Compliance</th>
<th>$P{J &gt; 5}$</th>
<th>$P{J &gt; 6}$</th>
<th>$P{J &gt; 7}$</th>
<th>$P{J &gt; 9}$</th>
<th>$P{J &gt; 10}$</th>
<th>$\sigma_J$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DTO</td>
<td>4.2393</td>
<td>0.7524</td>
<td>0.6269</td>
<td>0.5419</td>
<td>0.4224</td>
<td>0.3763</td>
<td>9.2215</td>
</tr>
<tr>
<td>$\eta = 5$</td>
<td>4.2422</td>
<td><strong>0.7093</strong></td>
<td>0.5274</td>
<td>0.4155</td>
<td>0.2772</td>
<td>0.2280</td>
<td>5.4176</td>
</tr>
<tr>
<td>$\eta = 6$</td>
<td>4.9628</td>
<td>0.9999</td>
<td><strong>0.4632</strong></td>
<td>0.2884</td>
<td>0.1299</td>
<td>0.0872</td>
<td>2.3927</td>
</tr>
<tr>
<td>$\eta = 7$</td>
<td>5.4498</td>
<td>1</td>
<td>0.4861</td>
<td><strong>0.2178</strong></td>
<td>0.0614</td>
<td>0.0311</td>
<td>1.3829</td>
</tr>
<tr>
<td>$\eta = 9$</td>
<td>6.8351</td>
<td>1</td>
<td>1</td>
<td>0.6557</td>
<td><strong>0.0581</strong></td>
<td>0.0249</td>
<td>0.8515</td>
</tr>
<tr>
<td>$\eta = 10$</td>
<td>7.9444</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0.1093</td>
<td><strong>0.0225</strong></td>
<td>0.5725</td>
</tr>
</tbody>
</table>

Figure 6: PDFs calculated for the DTO design and RATO designs with different values of $\eta$ in the beam-to-cantilever benchmark.

When the compliance is close to such a threshold value, and thus the mean performance is degraded. This can lead to deterministic design showing lower levels of $P\{J > \eta_i\}$ compared to risk-averse designs obtained with $\eta > \eta_i$ values. This conclusion is only pertinent to the problem discussed in this section and, as shown in the following examples, is heavily dependent on the impact of uncertainty in the performance of the deterministic design. Finally, it should be noted that the increment of the mean of PDFs along with the reduction in the Excess Probability involve a reduction of the dispersion in the PDF. This fact can be observed in the last column of Table 1, where the standard deviation of the compliance is shown for the different cases analyzed. One can observe that the value of the standard deviation is reduced as the threshold $\eta$ increases. In addition to reduce the Excess Probability, the risk-averse formulation provides designs which are less sensitive to fluctuations and thus more robust. This fact explains the topological similarities between the risk-averse designs and the ones obtained in [13] using RTO.
5.2. Numerical Experiment 2: Carrier plate design

The carrier plate design problem [14] aims to show RATO of continuum structures under random field loading uncertainty. The uncertain loading, the boundary conditions and the tessellation of the design domain are shown in Figure 7(a), whereas the initialization of the level-set function is shown in Figure 7(b). The dimension of the domain is a square of side length 200. The domain is tessellated using a regular mesh of 128x128 linear quadrangular elements \( h_x = h_y = 1.5625 \). Three layers of elements at the top of the domain are defined as non-optimizable to ensure the structure remains attached to the loading conditions. The structure is subjected to a uniformly vertically distributed load \( g_2 = 10 \) acting at its top. Additionally, a random horizontal distributed load \( g_1(x, w) \) is applied at the top of the structure. The load magnitude is modeled as a Gaussian random field with zero mean and isotropic square exponential covariance function

\[
C(x, x') = \sigma^2 \exp \left[ -\sum_{i=1}^{2} \frac{(x_i - x'_i)^2}{l^2} \right],
\]

where \( x = (x_1, x_2) \), \( x' = (x'_1, x'_2) \) \( \in \Gamma_N \), \( \sigma = 0.05 \) is the standard deviation and \( l = 30 \) is the correlation length. Figure 8(a) shows the eigenvalues of the correlation matrix that determines the number of random variables for the carrier plate experiment. According to the decay rate of the eigenvalues, the K-L expansion is truncated at its fifth term capturing the 60% of the energy field.
The level of the PC expansion is computed adaptively as described in Section 4.2 with $\ell = 10$ and $\delta = 1e-2$. For a level $\ell = 5$, the maximum indexes $I_\beta(5,5) = \{5,4,4,3,3\}$ and the value $D_{KL} = 1.2e-3$ are obtained. Using the anisotropic sparse grid configuration, each iteration of the RATO requires to solve 1347 different occurrences of the state equation in contrast to the 2203 and the 7776 that are required with the isotropic Smolyak’s sparse grid and the full tensor product respectively. The accuracy of the stochastic approximation is shown in Figure 9, where the PDF obtained using PC expansion is compared to the obtained using an enriched level $\ell = 10$. One can observe in Figure 9 that the agreement is satisfactory.

Similarly to the previous example, the volume constraint is set to the 30% of the initial design domain. Figures 10(b-d) show the topologies obtained solving the RATO problem with thresholds $\eta = 620$, $\eta = 1000$ and $\eta = 1500$, respectively. The topologies are compared to the one obtained using a deterministic formulation, which is depicted in Figure 10(a). One can observe that the risk-averse designs differ meaningfully from the DTO design. The risk-averse designs show a larger span between supports than the deterministic optimal solution. Besides, the material distribution of stochastic designs is further from the symmetry axis than the deterministic optimal solution, which provides a higher lateral stiffness to such risk-averse designs. In the same vein, the number of members of the risk-averse designs is higher than the deterministic counterpart, which diversifies the load paths. Regarding the risk-averse designs, as the threshold $\eta$ increases the topologies evolve by changing the number of holes and the thickness of the members.

The deterministic optimization required 175 iterations to converge whereas the risk-averse optimization (for case $\eta = 620$) required 153 iterations to converge. Each iteration of the risk-averse optimization required to solve the 1347
Figure 9: The carrier plate design problem (case $\eta = 1000$): Accuracy of PDF estimation using PC expansion method.

Table 2: Excess Probability of optimal design for different values of $\eta$ and for DTO in carrier plate design problem.

<table>
<thead>
<tr>
<th>Case</th>
<th>Compliance</th>
<th>$\mathbb{P}{J &gt; 620}$</th>
<th>$\mathbb{P}{J &gt; 1000}$</th>
<th>$\mathbb{P}{J &gt; 1500}$</th>
<th>$\sigma_J$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DTO</td>
<td>335.855</td>
<td>0.9057</td>
<td>0.8511</td>
<td>0.8012</td>
<td>2.553e4</td>
</tr>
<tr>
<td>$\eta = 620$</td>
<td>410.364</td>
<td><strong>0.3051</strong></td>
<td>0.0728</td>
<td>0.0142</td>
<td>254.9993</td>
</tr>
<tr>
<td>$\eta = 1000$</td>
<td>458.509</td>
<td>0.3402</td>
<td><strong>0.0618</strong></td>
<td>0.0103</td>
<td>214.7807</td>
</tr>
<tr>
<td>$\eta = 1500$</td>
<td>525.013</td>
<td>0.5064</td>
<td>0.0790</td>
<td><strong>0.0085</strong></td>
<td>207.9096</td>
</tr>
</tbody>
</table>

different occurrences of the state equation, which were performed concurrently at each worker of the cluster. The linear system of equations of elasticity is solved using a V-cycle geometric multigrid preconditioned conjugate gradient method with five levels and a damping factor for Jacobi smoothing $w = 0.4$.

The time per iteration of the risk-averse optimization using 96 workers was 1.75 sec. whereas the deterministic optimization was 0.101 sec.

The Excess Probabilities for optimal deterministic and risk-averse designs are detailed in Table 2. Similarly to the previous experiment, the probability of exceeding a critical value $\eta_i$ achieves its minimal value when the quantity $\mathbb{P}\{J > \eta_i\}$ is considered as objective function. Such values are highlighted in bold. Contrary to previous experiment, the deterministic design provides the worst results for all the cases analyzed. This is attributed to the significant impact of horizontal random field uncertainty in the stiffness of the deterministic design. One can observe that the spreading of the PDF for the DTO design, shown in Figure 11(a), is significantly higher than its risk-averse counterpart, shown in Figure 11(b). Similarly to the previous experiment, the mean of PDFs is incremented as the prescribed threshold $\eta_i$ increases. This increment comes with a reduction of the standard deviation of the compliance as shown in the last
Figure 10: The carrier plate design problem: (a) DTO design and (b-d) RATO designs for different values of $\eta$. 

(a) DTO

(b) $\eta = 620$

(c) $\eta = 1000$

(d) $\eta = 1500$
column of Table 2. This behavior shows the capacity of risk-averse formulation to not only minimize the influence of extreme events but also to increase the robustness of the structure reducing the standard deviation of the response.

5.3. Numerical Experiment 3: Michell-type structure design

The Michell-type structure benchmark is used to analyze the risk-averse approach under combined loading and material uncertainties. The boundary conditions and the domain discretization are shown in Figure 12(a), whereas the initialization of the level-set function is shown in Figure 12(b). The design domain has a roller support in the bottom right-hand corner and a fixed support in the bottom left-hand corner. The domain is tessellated using a regular grid of 160x64 linear quadrangular elements \( h_x = h_y = 1.25 \). The structure is subjected to three uncertain concentrated loads applied at its bottom. The magnitude of the loads are characterized by three independent random variables. These random variables are assumed to follow normal distributions with mean values \( \mu_{g1} = \mu_{g2} = \mu_{g3} = 1 \) and standard deviations \( \sigma_{g1} = 0.5, \sigma_{g2} = 0.1 \) and \( \sigma_{g3} = 0.2 \). Additionally, the uncertainty in the Young’s modulus is modeled by a 2D lognormal random field with mean \( \mu_F = 1 \) and standard deviation \( \sigma_F = 0.3 \). Such a lognormal random field is obtained through the transformation

\[
F(x) = \exp(\eta + \xi U(x)),
\]  

(41)

with
\[ \eta = \log \left( \frac{\mu_\beta^2}{\sqrt{\mu_\beta^2 + \sigma_\beta^2}} \right) \]
\[ \xi = \left( \log \left( 1 + \frac{\sigma_\beta^2}{\mu_\beta^2} \right) \right)^{1/2} \]

(42)

where \( \eta \) and \( \xi \) are the location and scale parameters of the lognormal distribution and \( U \) is assumed to have zero mean, unit variance and the isotropic squared exponential covariance function (40), which is defined in the whole spatial domain. A correlation length \( l_c = 60 \) is considered in this problem. Figure 8(b) shows the eigenvalues of the correlation matrix that determines the number of random variables for the Michell-type experiment. According to the decay rate of the eigenvalues, the Gaussian random field \( U \) is discretized through the truncated K-L expansion using four terms, which captures the 55% of the energy field. The resulting stochastic domain is composed of 7 random variables, 3 of them characterizing the uncertain loads and 4 of them characterizing the material variability. A PC expansion is used to approximate the compliance in the stochastic domain.

The level of the PC expansion is computed adaptively as described in Section 4.2 with \( \ell = 10 \) and \( \delta = 1.0e - 2 \). For a level \( \ell = 4 \), the maximum indexes \( I_\beta(4, 7) = \{4, 4, 4, 4, 3, 2, 2\} \) and the value \( D_{KL} = 1.8e - 3 \) are obtained. Using the anisotropic sparse grid configuration, each iteration of the RATO requires to solve 2337 different occurrences of the state equation in contrast to the 2437
and the 78125 that are required with the isotropic Smolyak’s sparse grid and the full tensor product, respectively. The accuracy of the stochastic approximation is shown in Figure 13, where the PDF obtained using PC expansion is compared to the obtained using an enriched level $\ell = 10$. One can observe in Figure 13 that the agreement is satisfactory.

The RATO problem is solved with the threshold values $\eta = 100$, $\eta = 200$ and $\eta = 300$, obtaining the optimized designs shown in Figure 14(b-d) for a target volume of the 30% of the initial design domain. One can observe that the RATO designs show significant topological differences with respect to their deterministic counterpart. As the threshold value $\eta$ increases, the topologies evolve modifying the number of holes and increasing the thickness of the members. One also can observe that the material distribution is oriented to the left in order to withstand the force with the highest level of uncertainty.

The deterministic optimization required 116 iterations to converge whereas the risk-averse optimization (for case $\eta = 200$) required 121 iterations to converge. Each iteration of the risk-averse optimization required to solve the 2337 different occurrences of the state equation, which were performed concurrently at each worker of the cluster. The linear system of equations of elasticity is solved using a V-cycle geometric multigrid preconditioned conjugate gradient method with five levels and a damping factor for Jacobi smoothing $w = 0.4$. The time per iteration of the risk-averse optimization using 96 workers was 2.5 sec. whereas the deterministic optimization was 0.083 sec.

Figure 15 shows the PDFs of the compliance for the deterministic optimal design and the risk-averse designs with different values of $\eta$. As occurred in the previous experiments, the mean compliance for RATO designs is incremented as the prescribed threshold $\eta$ increases. The Excess Probability of compliance for optimal deterministic and risk-averse designs is also detailed in Table 3.
5.4. Discussion

The numerical experiments have shown the effectiveness of the proposal to address RATO problems involving smooth random fields not only in the loading source but also in material properties. The combination of Adaptive Anisotropic PC expansion and Sparse Grids permits to notably reduce the computational cost of problems involving random fields with a moderate number of random variables. Furthermore, the embarrassingly parallel nature of these techniques facilitates the use of HPC on parallel and distributed architectures.

Some conclusions can be drawn from the numerical results. Firstly, the minimization of the Excess Probability is accomplished at the cost of degrading the structural performance under nominal conditions. Secondly, despite the differences in formulation, the RATO and RTO show clear similitudes in their optimal results. For all the experiments carried out in this work, the minimization of the risk of exceeding a prescribed threshold is accompanied by an increment of the structural robustness. However, whereas in RTO the robust designs are obtained modifying a weighting factor that balances the expected value and the standard deviation [13], which is subjectively selected a priori by the designer, in RATO the optimal designs are controlled by a parameter \( \eta \), which is related to the structural performance, and hence, it is a more objective measure.
Figure 15: The Michell-type structure design problem: (a) PDFs of DTO design and RATO designs with different values of $\eta$, and (b) detailed view of the PDF for values of compliance higher than 200.

Table 3: The Michell-type structure design problem: Excess Probability of DTO design and RATO designs with different values of $\eta$.

<table>
<thead>
<tr>
<th>Case</th>
<th>Compliance</th>
<th>$P{ J &gt; 100 }$</th>
<th>$P{ J &gt; 200 }$</th>
<th>$P{ J &gt; 300 }$</th>
<th>$\sigma_J$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DTO</td>
<td>104.218</td>
<td>0.6278</td>
<td>0.0516</td>
<td>0.0015</td>
<td>43.8265</td>
</tr>
<tr>
<td>$\eta = 100$</td>
<td>104.412</td>
<td><strong>0.6198</strong></td>
<td>0.0554</td>
<td>0.0019</td>
<td>44.8327</td>
</tr>
<tr>
<td>$\eta = 200$</td>
<td>106.033</td>
<td>0.6563</td>
<td><strong>0.0459</strong></td>
<td>0.0017</td>
<td>41.5052</td>
</tr>
<tr>
<td>$\eta = 300$</td>
<td>117.812</td>
<td>0.7969</td>
<td>0.0660</td>
<td><strong>0.0010</strong></td>
<td>40.6750</td>
</tr>
</tbody>
</table>
6. Conclusion

This work has presented an adaptive anisotropic stochastic expansion framework to address the risk-averse structural topology optimization problem in the continuous weak form of the linear system of elasticity with random fields. A key difficulty is that the risk-averse topology optimization process makes use of the analytic expressions for the continuous shape derivatives, which require the integration over failure regions. This issue is addressed both from the theoretical and numerical viewpoints. The proposed RATO approach provides a general framework that is neither limited to the use of discrete random variables nor the discretization method. Theoretically, the proper stochastic problem formulation is ensured by the proof of the existence of an optimal shape on the set of admissible shapes. Numerically, the Excess Probability is efficiently computed by combining a non-intrusive anisotropic PC approach with a MC sampling method for uncertainty propagation. From a computational point of view, the anisotropic nature of the K-L expansion is exploited to reduce the number of coefficients and stochastic collocation points required by the stochastic expansion. Additionally, the non-intrusive nature of the method is preserved enabling the use of task-level parallelism to address the computational burden of RATO problem. The effectiveness of the proposed approach is demonstrated solving benchmarks subjected to loading and material uncertainties, including random variables and smooth random fields. The results show that risk-averse and robust formulations have strong similitudes. However, in RATO formulations the optimized design only depends on the threshold $\eta$, which is related with the structural performance, and hence, its choice is less arbitrary than the weighting factor in RTO.

7. Acknowledgements

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