Robust Optimal Design of structures via Kriging models

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Outline

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   Motivation
   Formulation
   Uncertainty quantification (UQ)

2. Kriging-based Robust Design Optimization
   Meta-modelling
   Kriging models
   Proposed approach

3. Numerical application
   Four-bar truss structure
   Kriging models
   Pareto frontier

4. Conclusion
1 Robust Optimal Design (ROD)
   - Motivation
   - Formulation
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   - Four-bar truss structure
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   - Pareto frontier

4 Conclusion
The optimization under uncertainty aims to obtain optimal designs less sensitive to the uncertainties inherent to the structural parameters.

**Sources of uncertainty:**
- Applied loads
- Spatial positions of joints
- Section properties
- Material properties
- Environmental conditions.
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Deterministic Optimal Design (DOD)

$$\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad f(x) \\
\text{s.t.} & \quad g_i(x) \leq 0 \quad i = 1, \ldots, m_i \\
& \quad h_j(x) = 0 \quad j = 1, \ldots, m_j \\
& \quad x_{\text{lower}} \leq x \leq x_{\text{upper}}
\end{align*} \quad (1)$$

Robust Optimal Design (ROD)

$$\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad \{\mu_{f(x,z)}(x), \sigma_{f(x,z)}(x)\} \\
\text{s.t.} & \quad \mu_{g_i(x,z)}(x) + \beta_i \sigma_{g_i(x,z)}(x) \leq 0 \quad i = 1, \ldots, m_i \\
& \quad \sigma_{h_j(x,z)}(x) \leq \sigma^+_j \quad j = 1, \ldots, m_j \\
& \quad x_{\text{lower}} \leq x \leq x_{\text{upper}}
\end{align*} \quad (2)$$

$x \equiv$ vector of design variables, $z \equiv$ vector of random parameters
The $k$th statistical moments of the structural performance can be analytically expressed using a multi-dimensional integral (3).

$$E\{g^k(x)\} = \int_{\Omega} g^k(x) \cdot f_X(x) \cdot dx,$$

(3)

The main challenge is how to solve the multidimensional integration.

UQ (Monte-Carlo)
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Meta-modelling

Consist of replacing a computationally expensive simulation model by a mathematical approximation which is much faster to evaluate.

1. To sample the function to be predicted.
2. To create a mathematical approximation using statistical considerations.
3. To evaluate the accuracy of the mathematical model.
Kriging models

Kriging models assume that the simulator can be approximated by a sample path of a Gaussian stochastic process $\mathcal{G}(\mathbf{x})$

**Prior mean**

$$E[\mathcal{G}(\mathbf{x})] = \mathbf{f}(\mathbf{x})^T \beta$$

**Prior covariance function**

$$\text{Cov}[\mathcal{G}(\mathbf{x}), \mathcal{G}(\mathbf{x}')] = \alpha^2 \exp\left(\sum_{i=1}^{n} - \frac{|x_i - x_i'|^s}{\phi_i}\right)$$

The parameters $\beta$, $\alpha^2$ and $\phi$ are unknown a priori and are determined from the set of simulator responses

$$\hat{\mathcal{Y}}(\mathbf{x}) \equiv E[\mathcal{G}(\mathbf{x})|\mathcal{Y}] = \mathbf{f}(\mathbf{x})^T \hat{\beta} + \mathbf{r}(\mathbf{x})^T \mathbf{R}^{-1}(\mathbf{Y}^T - \mathbf{F}\hat{\beta})$$
Proposed approach

Flowchart

Step 1: global approximation
\( f(x, z) \)

- Generation of DoE
\( \Xi = \{x_1z_1, \ldots, x_mz_m \} \)

- Fit Kriging models in \( x, z \) space (maximize likelihood)

- Identify the best Kg model (Best \( PRESS_{MSE} \))

- Accurate models? (Predicted \( R^2 \))

- Accurate models?

Step 2: Uncertainty Quantification

- Uncertainty quantification assisted by Kriging models

- \( \tilde{\mu}_f(x, z) \)
- \( \tilde{\sigma}_f(x, z) \)
- \( \tilde{\beta}_f(x, z) \)
- \( \tilde{\gamma}_f(x, z) \)

Step 3: global approximation of statistical moments

- Fit Kriging models in \( x \) space (maximize likelihood)

- Identify the best Kg model (Best \( PRESS_{MSE} \))

- Accurate models? (Predicted \( R^2 \))

- Accurate models?

Step 4: optimization

- Optimization using MOEA (NSGA-II)

- Validation of Robust optimal solutions

- Accurate solutions?

End
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Problem definition

four-bar truss structure

(Dolsitis et al., 2004)

\[ \begin{align*}
\min_{A_1, A_2} & \{ \tilde{\mu}(A_1, A_2, E_1, E_2)(A_1, A_2), \tilde{\sigma}(A_1, A_2, E_1, E_2)(A_1, A_2) \} \\
\text{s.t. } & w \leq 5 \\
& 0 \leq A_{1,2} \leq 2
\end{align*} \]

Design variables:
- \( A_1 \) (bars 1 and 3).
- \( A_2 \) (bars 2 and 4).

Random parameters:
- \( E_1 \sim \mathcal{N}(210, 21) \) (bars 1 and 3).
- \( E_2 \sim \mathcal{N}(100, 15) \) (bars 2 and 4).
Problem definition

four-bar truss structure

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- $E_1 \sim \mathcal{N}(210, 21)$ (bars 1 and 3).
- $E_2 \sim \mathcal{N}(100, 15)$ (bars 2 and 4).
The Kriging model was created based on 100 pieces of information obtained from an Optimized Latin Hypercube Sampling, which was improved using 100 additional infill samples.

True function (simulator) \( u(A_1, A_2, E_1, E_2) \)

Kriging model \( \tilde{u}(A_1, A_2, E_1, E_2) \)
The statistical moments were obtained by Monte Carlo simulation for a sample size of 10000.

\[ \tilde{\mu}(A_1, A_2, E_1, E_2)(A_1, A_2) \]

\[ \tilde{\sigma}(A_1, A_2, E_1, E_2)(A_1, A_2) \]
Multi-objective optimization
Robust Pareto fronts & feasible region

Robust Pareto frontier

\[ \tilde{\mu}_u(A_1, A_2, E_1, E_2) (A_1, A_2) \]

\[ \tilde{\sigma}_u(A_1, A_2, E_1, E_2) (A_1, A_2) \]
Multi-objective optimization

Accuracy

\[ \min \sigma_u(x) \]

\[ \min \mu_u(x) \]

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<th>Real</th>
<th>Error(%)</th>
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<td>( \mu_u(x) )</td>
<td>3.965E-3</td>
<td>3.967E-3</td>
</tr>
<tr>
<td>( \sigma_u(x) )</td>
<td>1.721E-4</td>
<td>1.786E-4</td>
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<tr>
<td>( \mu_u(x) )</td>
<td>3.843E-3</td>
<td>3.846E-3</td>
</tr>
<tr>
<td>( \sigma_u(x) )</td>
<td>4.007E-4</td>
<td>3.961E-4</td>
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Concluding remarks

Conclusions:

1. The proposed approach has shown to be efficient in low-dimensional problems that involved computationally demanding simulation models.

2. The global approximation allow us not only to speed up the optimization algorithm, but also to explore the design space, improving the formulation of the problem.

3. The Kriging models can be re-used in new optimization processes or computationally demanding applications.

Future works:

1. High-dimensional structural problems.
Thanks for your attention!

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